Basics of the GPS Technique: Observation Equations

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Topics of Lecture

- GPS description
- Pseudorange observable
  - point positioning
- Carrier phase observable
  - differencing techniques and relative positioning
  - advanced observable model (undifferenced)
- Analytical observable model and algorithms

GPS Description

- The basic idea
- GPS Segments
  - Space, Control, User, and Ground
- GPS signals
  - Code Generation
  - Navigation Message
  - Denial of Accuracy
    - Selective Availability (S/A)
    - Anti-Spoofing (A/S)

The Basic Idea

- GPS positioning is based on trilateration
- Trilateration
  - 3 ranges to 3 known points
- GPS point positioning
  - 4 "pseudoranges" to 4 satellites
  - How do we know position of satellites?
  - What are pseudoranges?

The Basic Idea

- GPS point positioning
  - How do we know position of satellites?
    - Receiver can read the Navigation Message
      - is encoded on the GPS signal
      - includes orbit parameters (broadcast ephemeris)
    - Receiver can compute satellite coordinates (X,Y,Z)
      - Use the standard Ephemeris Algorithm

The Basic Idea

- GPS point positioning
  - What are pseudoranges?
    - Time of signal transmission
      - is encoded on signal by the satellite using an atomic clock
    - Time of signal reception
      - is recorded by receiver using an atomic clock
    - Receiver measures difference in these times
      - pseudorange = (time difference) × (speed of light)
    - Pseudorange is like range, but includes clock errors

The Basic Idea

- GPS point positioning
  - How do we correct for clock errors?
    - Satellite clock error is given in Navigation Message
    - Receiver clock error is computed with coordinates
    - 4 unknowns require at least 4 observations
      - 3 receiver coordinates + 1 receiver clock error
      - GPS positioning requires 4 pseudoranges

GPS Segments

- 4 segments:
  - Space Segment
    - Constellation of satellites
  - Control Segment
    - Monitor and operation of satellites and signals
  - User Segment
    - User hardware and processing software
  - Ground Segment
    - Civilian networks of reference stations

Space Segment

- Satellite constellation design
  - At least 4 satellites in view, anywhere, anytime
  - Nominally 24 GPS satellites
    - 6 orbital planes
      - a Semi-major axis approximately 26,600 km
      - e Eccentricity 0.02 (nearly circular)
      - Ω Ascending Nodes of each plane spaced by 60°
      - i Inclination 55°
    - 4 satellites per plane
      - M Mean anomoly spaced by approximately 90°

Space Segment

- Orbit design consequences
  - Satellite speed
    - approximately 4 km/s relative to Earth center
    - up to 3 km/s in direction of user
  - Repeating ground tracks
    - Orbital period T = 11 hr 58 min
    - But: Earth rotates 360° in 23 hr 56 min
      - Precisely 2 orbits per sidereal day
      - Therefore ground tracks repeat

Space Segment

- Orbit design consequences
  - Satellite geometry viewed by user
    - Satellite geometry repeats every sidereal day
      - Sky tracks repeat every (24 hours – 4 minutes)
      - Same subset of satellites viewed every day
      - any errors correlated with satellite geometry will repeat from one day to the next
    - Inclination = 55 degrees
      - At the poles, no satellite can rise above 55° elevation
      - At equator, satellites track approximately
        - North to South, or South to North
        - Little East-West motion

Control Segment

- Responsible for operating GPS
- Structure
  - Operated by US Air Force
  - Control Centre: Falcon Air Force Base, Colorado Springs
  - Several ground sites monitor satellites:
    - assess satellite health
    - estimate satellite orbits and clocks
    - predict satellite orbits and clocks

GPS Signal

- Driven by atomic clock
  - Frequency = 10.23 MHz
- Two carrier signals (sine waves)
  - L1 = 154 x 10.23 MHz wavelength = 19.0 cm
  - L2 = 120 x 10.23 MHz wavelength = 24.4 cm
- Bits encoded on carrier by phase modulation
  - C/A code
  - P code
  - Navigation Message

GPS Signal

- Course Acquisition (C/A) Code
  - Bit sequence
    - Repeats every 1 ms
    - Pseudo-random (appears random, but sequence known)
    - 1.023 Mbps (million bits per second)
    - Chip length = 293 metres
  - Information
    - Satellite clock time

GPS Signal

- Precise (P) Code
  - Bit sequence
    - Repeats every 267 days
    - Pseudo-random (appears random, but sequence known)
    - 10.23 Mbps (million bits per second)
    - Chip length = 29.3 metres
  - Information
    - Satellite clock time
  - Can be encrypted
    - Anti Spoofing (A/S)

GPS Signal

- Navigation Message
  - Bit sequence
    - 1500 bit sequence at 50 bps
    - 30 seconds to transmit
  - Information
    - Broadcast Ephemeris (S/A)
    - Satellite Clock Corrections (S/A)
    - Almanac Data
    - Ionosphere Information
    - Satellite Health

GPS Signal

- Selective Availability (S/A)
  - Intentional errors imposed on the GPS signal
  - Epsilon
    - Navigation message includes errors in ephemeris
    - apparently not used (according to daily comparisons between broadcast and IGS orbit solutions)
    - Has no effect when using precise orbits
  - Dither
    - the satellite reference frequency is dithered
    - looks exactly like a satellite clock error
  - S/A is mitigated by relative positioning

GPS Signal

- Anti-Spoofing (A/S)
  - Encryption of the P-code
  - Modern geodetic receivers can nevertheless form 2 precise pseudorange observables
  - Pseudorange noise is increased
    - However, we rely on phase for precise work
    - May degrade cycle-slip detection and ambiguity resolution
    - L2 phase noise is increased
      - from ~1 mm to ~ 1 cm
      - No serious effect on long sessions (static positioning)
      - Degrades short session positions (rapid static, kinematic)
      - Larger degradation at low elevations (up to 2 cm)

The Pseudorange Observable

- How is the transmitted signal encoded?
  - Pseudorandom (PRN) code generation
    - XOR binary function
    - Linear feedback registers
  - C/A code
  - P code
- How is the pseudorange observable formed
  - Discrete autocorrelation technique

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C/A Code

- 10 stage linear feedback register sequence
  - Sequence length $L(10) = 2^{10}-1 = 1023$
- C/A has code repeating sequence of 1023 bits
  - which appear to be random
- C/A bit transitions occur at 1.023 MHz
  - Sequence repeats 1000 times per second
  - Time for each sequence is 1 ms
- Chip length (between bits) is 293 metres
  - Sequence repeats every 300 km
  - "Ambiguity" in the C/A code
    - easy to acquire

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P Code

- Generated from a combination of two different registers
- P code sequence repeats in 266.4 days
  - Each 7 day section is assigned a PRN code
    - Satellites often identified by their PRN number
    - PRN 2 refers to week 2 of the sequence
    - There are 38 possible PRN codes
      - 24 satellites
      - some PRN codes are unused
    - PRN sequence is reset at Saturday midnight
      - start of GPS week

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Autocorrelation

- Replica Signal
  - Receivers generate same codes as the satellites
- Pseudorange Measurement
  - receiver matches incoming signal with replica signal
  - The time difference is computed by autocorrelation
    - Multiply bits from signal and replica signal and sum
    - The replica signal is shifted one bit at a time until this number is maximized
    - This is why the codes are designed to look random
  - The result is multiplied by the speed of light
    - This measurement is called the pseudorange

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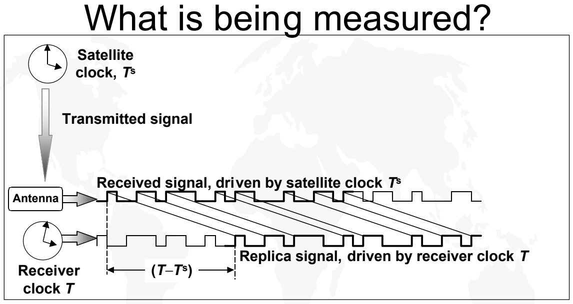
What is being measured?



- shift bits of replica signal until it matches received signal

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Point Positioning

- Pseudorange observation model
- Observation equations
- Linearised observation model
- Least squares estimation
- What are "good" conditions?
  - Dilution of Precision (DOP)

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Observation Model

- Actual observation

$$P_r^s = (T_r - T^s)\, c$$

  - at receiver$_r$
    - whose clock reads T_r when signal is received
  - from satellite^s
    - whose clock read T^s when transmitted
    - c is the speed of light (in a vacuum)

Observation Model

- Modelled observation
  - denote clock times by T, and true times by t

$$\begin{aligned} P_r^s &= (T_r - T^s)c \\ &= (t_r + \delta t_r - t^s - \delta t^s)c \\ &= (t_r - t^s)c + c\delta t_r - c\delta t^s \\ &= \rho_r^s + c\,\delta t_r - c\delta t^s \end{aligned}$$

  - where

ρ_r^s	range from receiver to satellite
δt_r	receiver clock error
δt^s	satellite clock error
c	speed of light = 299792458 m/s

  - this model is simplified
    - since it assumes the speed of light in the atmosphere is c

Observation model

$$P_r^s = \rho_r^s + c\,\delta t_r - c\,\delta t^s$$

- Pythagoras Theorem:

$$\rho_r^s = \left((u^s - u_r)^2 + (v^s - v_r)^2 + (w^s - w_r)^2 \right)^{1/2}$$

- Knowns: Navigation message gives us
  - satellite position (u^s, v^s, w^s)
  - satellite clock error δt^s can be used to correct P_r^s
- 4 unknowns:
  - receiver position (u_r, v_r, w_r)
  - receiver clock error δt_r

Observation Equations

- Consider:
  - 4 satellites in view of receiver “A”
    - Receiver index: r = A
    - Satellite index: s = 1, 2, 3, 4

$$P_A^1 = \left((u^1 - u_A)^2 + (v^1 - v_A)^2 + (w^1 - w_A)^2 \right)^{1/2} + c\,\delta t_A$$

$$P_A^2 = \left((u^2 - u_A)^2 + (v^2 - v_A)^2 + (w^2 - w_A)^2 \right)^{1/2} + c\,\delta t_A$$

$$P_A^3 = \left((u^3 - u_A)^2 + (v^3 - v_A)^2 + (w^3 - w_A)^2 \right)^{1/2} + c\,\delta t_A$$

$$P_A^4 = \left((u^4 - u_A)^2 + (v^4 - v_A)^2 + (w^4 - w_A)^2 \right)^{1/2} + c\,\delta t_A$$

Linearised Observation Model

- Consider observation P, a function of parameters u, v, w, ...
- Let P_0 denote P computed using provisional values u_0, v_0, w_0, ...
- Taylor expansion is:

$$P(u,v,\ldots) = P(u_0,v_0,\ldots) + (u-u_0)\frac{\partial P}{\partial u} + (v-v_0)\frac{\partial P}{\partial v} + \ldots$$

$$\Delta P = \frac{\partial P}{\partial u}\Delta u + \frac{\partial P}{\partial v}\Delta v + \ldots$$

$$\Delta P = \begin{pmatrix} \frac{\partial P}{\partial u} & \frac{\partial P}{\partial v} & \ldots \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \Delta P^1 \\ \Delta P^2 \\ \Delta P^3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{\partial P^1}{\partial u} & \frac{\partial P^1}{\partial v} & \ldots \\ \frac{\partial P^2}{\partial u} & \frac{\partial P^2}{\partial v} & \ldots \\ \frac{\partial P^3}{\partial u} & \frac{\partial P^3}{\partial v} & \ldots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \vdots \end{pmatrix}$$

Least Squares Solution

- Linearised model:

$\mathbf{b} = \mathbf{A}\mathbf{x} + \nu$ where $\nu =$ noise

$\mathbf{b} =$ column matrix of observations
  = observed P − computed P_0

$\mathbf{x} =$ column matrix of parameters
  = adjustment to provisional values: $(u\text{-}u_0)$, $(v\text{-}v_0)$, etc.

$\mathbf{A} =$ square design matrix (partial derivatives $\partial P/\partial u$, etc.)

- Least squares solution:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Covariance Matrix and Errors

$$Q_{\mathbf{x}} = \left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1} = \begin{pmatrix} \sigma_u{}^2 & \sigma_{uv} & \sigma_{uw} & \sigma_{ut} \\ \sigma_{uv} & \sigma_v{}^2 & \sigma_{vw} & \sigma_{vt} \\ \sigma_{uw} & \sigma_{vw} & \sigma_w{}^2 & \sigma_{wt} \\ \sigma_{ut} & \sigma_{vt} & \sigma_{wt} & \sigma_\tau{}^2 \end{pmatrix}$$

- Interpretation example:
  - If observation errors are 2 metres, then
    - error in u coordinate is $2\sigma_u$ metres
  - Off diagonal elements indicate degree of correlation
    - ☛If σ_{uv} is negative, this means that a positive error in u will probably be accompanied by a negative error in v
  - Elements $\sigma_u, \sigma_v, \sigma_{uv}$ define an "error ellipse"

Local Coordinate Errors

- It is more convenient to look at errors in local geodetic coordinates
  - Transform geocentric *(u,v,w)* to local *(n, e, h)*
  - Applications tend to focus on horizontal and vertical
  - Also, height, h, tends to have largest error
- Method: Law of error propogation
  - Let us define transformation matrix *G* which takes us from small relative vector in geocentric system into local system:

$$\Delta L = (\Delta n, \Delta e, \Delta h) = G\,\Delta X$$

Local Coordinate Errors

$$Q_{uvw} = \begin{pmatrix} \sigma_u{}^2 & \sigma_{uv} & \sigma_{uw} \\ \sigma_{uv} & \sigma_v{}^2 & \sigma_{vw} \\ \sigma_{uw} & \sigma_{vw} & \sigma_w{}^2 \end{pmatrix}$$

$$Q_{neh} = G Q_{uvw} G^T = \begin{pmatrix} \sigma_n{}^2 & \sigma_{ne} & \sigma_{nh} \\ \sigma_{ne} & \sigma_e{}^2 & \sigma_{eh} \\ \sigma_{nh} & \sigma_{eh} & \sigma_h{}^2 \end{pmatrix}$$

- Result is a covariance matrix in local system
- Can look up transformation matrix G
  - Is a function of longitude, latitude and height
  - You also need ellipsoidal parameters *a* and *e*.

Dilution of Precision

$$VDOP \equiv \sigma_h$$

$$HDOP \equiv \sqrt{\sigma_n^2 + \sigma_e^2}$$

$$PDOP \equiv \sqrt{\sigma_n^2 + \sigma_e^2 + \sigma_h^2} + \sqrt{\sigma_u^2 + \sigma_v^2 + \sigma_w^2}$$

$$TDOP \equiv \sigma_t$$

$$GDOP \equiv \sqrt{\sigma_u^2 + \sigma_v^2 + \sigma_w^2 + c^2\sigma_t^2}$$

- VDOP: Vertical Dilution of Precision
- HDOP: Horizontal Dilution of Precision
- PDOP: Position Dilution of Precision
- TDOP: Time Dilution of Precision
- GDOP: Geometric Dilution of Precision

Dilution of Precision

- Interpretation examples:
  - 1 metre error in observations
    - gives HDOP metres error in horizontal position
    - gives TDOP seconds error in receiver clock time
  - Good geometry
    - Small DOP is good, large DOP is bad
    - Values of HDOP and VDOP should be less than 5.
    - Values of 2 is typical if there are 5 or more satellites in view
    - Small DOP's achieved if satellites are well spread out
    - If two satellites pass close in the sky
      - you effectively loose a satellite in this calculation.
      - if only 4 satellites in view, DOP can get too large

Mission Planning

- Measures of good conditions:
  - Satellite geometry
    - Is a function of geographic location
    - Number of satellites visible
    - Dilution of precision
  - Elevation mask
    - elevation cutoff in receiver
    - elevation cutoff in software
    - physical obstructions
  - Low multipathing environment

The Carrier Phase Observable

- Concepts
  - Meaning of "phase"
  - Principle of interferometry
    - Doppler effect
  - Carrier phase observation
    - Dealing with the codes
  - Observation model
    - Phase ambiguity and cycle slips
    - Range formulation

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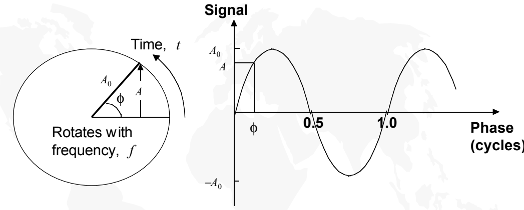
Meaning of Phase, ϕ



- Steady rotation
  $\phi = f\,t$ — phase ϕ (cycles), frequency f (Hz)
- Sine wave:
  $A = A_0 \sin(2\pi\,\phi)$ — amplitude $A_0 = A_0 \sin(\omega t)$
  angular frequency $\omega = 2\pi f$

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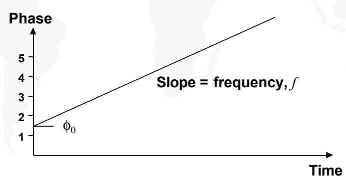
Meaning of Phase

- Therefore, for a sine wave
  - phase changes linearly in time:

    $\phi = f\,t$ — phase ϕ (cycles), frequency f (Hz)
- Generally, the initial phase is not zero:

  $\phi = f\,t + \phi_0$ — constant: ϕ_0 (cycles)



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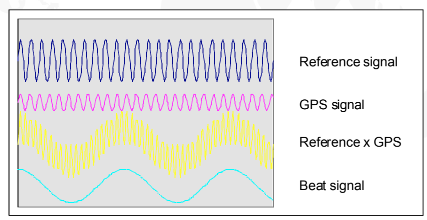
Carrier Beat Signal

- Multiply together 2 signals of similar frequency
- Result is sum of high + low frequency signals
- Filter out the high frequency, leave low frequency
  - "beat frequency" = difference in frequency of two signals
  - "beat phase" = difference in phase of two signals



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Doppler Effect

- Imagine two perfect clocks
  - One is at a fixed point, the other is approaching in a vehicle
  - Both clocks generate a sine-wave signal
  - Frequency difference increases with speed
- Build a receiver to mix the two signals
  - the beat frequency measures the speed
- Count the cycles of the beat signal
  - Better yet: measure the phase of the beat signal
  - this measures the change in distance to vehicle
    - Beat phase = distance to vehicle + constant

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Carrier Phase Observation

- Signal Mixing
  - The satellite carrier signal (from antenna) is mixed with reference signal generated by receiver's clock.
- The result is a "beating" signal.
  - Phase of this beating signal equals: reference phase – satellite carrier phase.
  - "Carrier beat phase" is often called "Carrier phase"

Observation of satellite j by receiver A

$$\varphi_A^{\,j} = \Phi_A - \Phi^j$$

Φ_A = reference phase generated by receiver A
Φ^j = signal phase received from satellite j

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Dealing with the Codes

- Code tracking
  - C/A code and P code is first removed from incoming signal (since they are known)
- Anti-Spoofing (A/S)
  - C/A code is known, hence L1 carrier phase can be measured
  - But P code is not known
- Squaring
  - Squaring incoming L2 wave eliminates the P-code
  - But it doubles the frequency, and increases noise
- Cross correlation
  - P code is the same on L1 and L2 frequencies
  - ñ Therefore, L2 signal is correlated with the L1 signal to produce an (L1−L2) carrier phase.

Observation Model

- Observation of satellite j by receiver A when receiver clock reads T_A

$$\varphi_A{}^j(T_A) = \Phi_{0,A} - \Phi_{GPS,A}{}^j$$

- where:

$\Phi_{0,A}$ = reference phase (cycles) generated by receiver clock, at receiver clock time T_A

$\Phi_{GPS,A}{}^j$ = GPS signal phase (cycles) as received from satellite j at receiver clock time T_A

  - GPS signal phase can be considered constant as it travels from satellite to receiver:

$\Phi_{GPS,A}{}^j = \Phi^{GPS,j}{}_A$ = GPS signal phase (cycles) as transmitted to receiver A at satellite clock time T^j

Phase Ambiguity

– BUT receiver does not know the integer component of GPS signal phase

- Therefore, we model:

$$\Phi_{GPS,A}{}^j = \Phi^{GPS,j}{}_A + N_A{}^j$$

  - where $N_A{}^j$ = integer ambiguity for satellite j (unknown to receiver A) = constant if receiver maintains lock
- Ambiguity resolution ("bias fixing"):
  - The problem of finding the correct integer ambiguity
- Cycle slips fixing:
  - Discontinuities in $N_A{}^j$ when receiver loses lock

Observation Model

- Carrier Phase Model
  - "beat phase"; "one-way phase"; "undifferenced phase"

$$\begin{aligned}\varphi_A{}^j(T_A) &= \Phi_{0,A} - \Phi_{GPS,A}{}^j \\ &= (f_0 T_A + \alpha_A) - (f_0 T^j + \alpha^j + N_A{}^j) \\ &= f_0(T_A - T^j) - (N_A{}^j + \alpha^j - \alpha_A)\end{aligned}$$

  - First term can be expressed as a pseudorange
  - Second term called is "Carrier Phase Bias" (constant)
    - Different carrier phase bias for each station-satellite pair
    - Although not an integer, it changes by an integer if receiver loses lock on signal ("cycle slip")

Range Formulation

– It is convenient to convert the carrier phase model into units of range.

– Simplifies concepts, models, and software.

– Range formulation: × nominal wavelength

$$\begin{aligned}L_A{}^j &= \lambda_0 \varphi_A{}^j \qquad \lambda_0 = c / f_0 \\ &= c(T_A - T^j) - \lambda_0(N_A{}^j + \alpha^j - \alpha_A) \\ &= c(T_A - T^j) - B_A{}^j\end{aligned}$$

$c(T_A - T^j)$ = pseudorange term

$B_A{}^j$ = carrier phase bias

Relative Positioning and Differencing Techniques

- Observation Equation
- Differencing
  - single
  - double
  - triple

Observation Equation

– Carrier phase (metres)

$L_A^j = \lambda_0 \varphi_A^j \qquad \lambda_0 = c / f_0$

$\quad = c\,(T_A - T^j) - B_A^j$

- ✦ $c\,(T_A - T^j)$ = pseudorange term
- ✦ $B_A^j = \lambda_0 (N_A^j + \alpha^j - \alpha_A)$ = constant

– Recall pseudorange:

$P_A^j = c\,(T_A - T^j)$

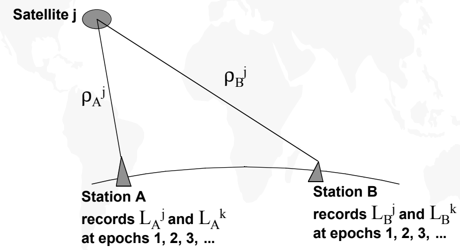
- ✦ which can be modelled:

$= \rho_A^j + \tau_A^j + c\,\delta t_A - c\,\delta t$

◆ Hence:

$$L_A^j = \rho_A^j + \tau_A^j + c\,\delta t_A - c\,\delta t^j - B_A^j$$

Single Differencing



Single Differencing

◆ Eliminate satellite clock error

– Two receivers observing same satellite, j

$L_A^j = \rho_A^j + \tau_A^j + c\,\delta t_A - c\,\delta t^j - B_A^j$

$L_B^j = \rho_B^j + \tau_B^j + c\,\delta t_B - c\,\delta t^j - B_B^j$

– Difference data between receivers:

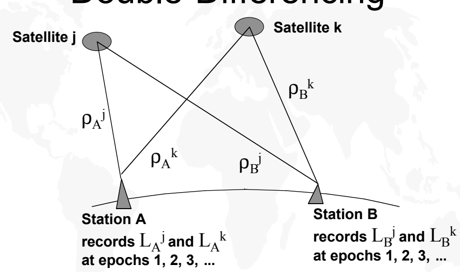
$\Delta L_{AB}^j = L_A^j - L_B^j$

$\quad = (\rho_A^j + \tau_A^j + c\,\delta t_A - c\,\delta t^j - B_A^j) -$

$\quad (\rho_B^j + \tau_B^j + c\,\delta t_B - c\,\delta t^j - B_B^j)$

$\Delta L_{AB}^j = \Delta\rho_{AB}^j + \Delta\tau_{AB}^j + c\,\Delta\delta t_{AB} - \Delta B_{AB}^j$

Double Differencing



Model: $L_{AB}^{jk} = (L_A^j - L_B^j) - (L_A^k - L_B^k)$

$\quad = \rho_{AB}^{jk} - \lambda_0 N_{AB}^{jk}$

Double Differencing

◆ Eliminate receiver clock error

– Two single differences from same station pair, but different satellites

$\Delta L_{AB}^j = \Delta\rho_{AB}^j + \Delta\tau_{AB}^j + c\,\Delta\delta t_{AB} - \Delta B_{AB}^j$

$\Delta L_{AB}^k = \Delta\rho_{AB}^k + \Delta\tau_{AB}^k + c\,\Delta\delta t_{AB} - \Delta B_{AB}^k$

– Difference data between receivers:

$\nabla\Delta L_{AB}^{jk} = \Delta L_{AB}^j - \Delta L_{AB}^k$

$\quad = (\Delta\rho_{AB}^j + \Delta\tau_{AB}^j + c\,\Delta\delta t_{AB} - \Delta B_{AB}^j) -$

$\quad (\Delta\rho_{AB}^k + \Delta\tau_{AB}^k + c\,\Delta\delta t_{AB} - \Delta B_{AB}$

$\nabla\Delta L_{AB}^{jk} = \nabla\Delta\rho_{AB}^{jk} + \nabla\Delta\tau_{AB}^{jk} - \nabla\Delta B_{AB}^{jk}$

Double Differenced Ambiguity

◆ Double difference ambiguity is integer

$\nabla\Delta B_{AB}^{jk} = \Delta B_{AB}^j - \Delta B_{AB}^k$

$\quad = (B_A^j - B_B^j) - (B_A^k - B_B^k)$

◆ Recall:

$B_A^j = \lambda_0 (N_A^j + \alpha^j - \alpha_A)$

◆ Result:

$\nabla\Delta B_{AB}^{jk} = \lambda_0(N_A^j - N_B^j) - \lambda_0(N_A^k - N_B^k)$

$\quad = \lambda_0 \nabla\Delta N_{AB}^{jk}$

$\nabla\Delta L_{AB}^{jk} = \nabla\Delta\rho_{AB}^{jk} + \nabla\Delta\tau_{AB}^{jk} - \lambda_0 \nabla\Delta N_{AB}^{jk}$

Relative Positioning using Double Difference Phase

- Observation Equation
- Linear dependence of observations
- Baseline solution
  - Weighted least squares
- Statistical dependence of observations
  - Double differenced data covariance
  - Stochastic model and the weight matrix

Observation Equation

- Double differenced carrier phase (metres)
  - $\nabla\Delta L_{AB}^{jk} \quad = \nabla\Delta\rho_{AB}^{jk} + \nabla\Delta\tau_{AB}^{jk} - \lambda_0\nabla\Delta N_{AB}^{jk}$
  - where:

    $\nabla\Delta\rho_{AB}^{jk} = (\rho_A^{j} - \rho_B^{j}) - (\rho_A^{k} - \rho_B^{k})$

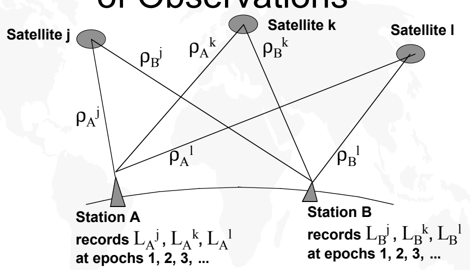
    $\rho_A^{j} = ((x^{j}-x_A)^2 + (y^{j}-y_A)^2 + (z^{j}-z_A)^2)$

    $\nabla\Delta\tau_{AB}^{jk} = \nabla\Delta$ atmospheric delay

    $\nabla\Delta N_{AB}^{jk} = \nabla\Delta$ phase ambiguity
  - For simplicity, from now on:
    - Assume $\nabla\Delta\ \tau_{AB}^{jk}$ is negligible
    - Drop the $\nabla\Delta$ notation
  - Observation equation for our purposes:

    $L_{AB}^{jk} \quad = \rho_{AB}^{jk} - \lambda_0 N_{AB}^{jk}$

Linear Dependence of Observations



- 3 Combinations:

  $L_{AB}^{jk} \quad = (L_A^{j} - L_B^{j}) - (L_A^{k} - L_B^{k})$

  $L_{AB}^{jl} \quad = (L_A^{j} - L_B^{j}) - (L_A^{l} - L_B^{l})$

  $L_{AB}^{lk} \quad = (L_A^{l} - L_B^{l}) - (L_A^{k} - L_B^{k})$

Linear Dependence of Observations

- Double differences can be linearly dependent
  - Note that: $L_{AB}^{lk} = L_{AB}^{jk} - L_{AB}^{jl}$
  - Therefore, data from L_{AB}^{lk} provides no new information
  - Similarly: $L_{AB}^{jk} = L_{AB}^{lk} + L_{AB}^{jl}$
    $L_{AB}^{jl} = L_{AB}^{jk} - L_{AB}^{lk}$
  - The set $\{L_{AB}^{jk}, L_{AB}^{jl}, L_{AB}^{lk}\}$ is linearly dependent
  - A linearly independent set is needed for least squares
  - Examples of linearly independent sets:
    - $(L_{AB}^{jk}, L_{AB}^{jl})$ = set $\Lambda^{j} = \{L_{AB}^{ab} \mid a{=}j,\ b{\neq}a\}$
    - $(L_{AB}^{kj}, L_{AB}^{kl})$ = set $\Lambda^{k} = \{L_{AB}^{ab} \mid a{=}k,\ b{\neq}a\}$
    - $(L_{AB}^{lj}, L_{AB}^{lk})$ = set $\Lambda^{l} = \{L_{AB}^{ab} \mid a{=}l,\ b{\neq}a\}$

Linear Dependence of Observations

- Reference satellite concept
  - One method of ensuring linear independence of data for baseline solution (between stations A and B)
  - Select a reference satellite.
  - For example, satellites 1, 2, 3, 4, and 5 are tracked.
  - Select reference satellite 3:

    $\Lambda^{3} \quad = \{L_{AB}^{jk} \mid j{=}3,\ k{\neq}3\}$

    $\quad = \{L_{AB}^{31}, L_{AB}^{32}, L_{AB}^{34}, L_{AB}^{35}\}$
  - All possible sets are valid
- Reference satellite selection
  - Select satellite which has longest data span
  - Better: select the best reference satellite every epoch

Linear Dependence of Observations

- Reference station/satellite concept
- For network solutions
  - between stations A, B, C, ...
  - Select a reference satellite and a reference station
  - For example, satellites 1, 2, 3, 4, and stations A, B, C
  - ñ Select reference satellite 4 and reference station A

    $\Lambda_A^{4} \quad = \{L_{ab}^{jk} \mid j{=}4,\ k{\neq}4,\ a{=}A,\ b{\neq}A\}$

    $\quad = \{L_{AB}^{41}, L_{AB}^{42}, L_{AB}^{43}, L_{AC}^{41}, L_{AC}^{42}, L_{AC}^{43}\}$
- Number of independent observations at each epoch

  $N_\Lambda \quad = (N_{stations}-1)(N^{satellites}-1)$
  - Example: 3 stations and 4 satellites give 6 observations

Baseline Solution

– Linearised model: $b = Ax + \nu$
- ν = noise
- b = column of observation residuals
  = observed (L_{AB}^{jk}) – computed ($\rho_{AB}^{jk} - \lambda_0 N_{AB}^{jk}$)
  – select a reference satellite j , and only include Λ^{j}
  ñ initially use provisional coordinates and N = 0
- x = column of parameters
  = adjustment to provisional values *x*, *y*, *z*, and N
  – If our estimate of N leads to an obvious integer, then fix this integer in the computed model ("bias fixing")
- A = design matrix (partial derivatives $\partial L/\partial u$, etc.)

– Weighted least squares solution:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{b}$$

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Statistical Dependence of Observations

- Double differences are also statistically dependent
  – L_{AB}^{12} and L_{AB}^{13} are correlated due to data in common L_A^1
  – An error in L_A^1 will affect both L_{AB}^{12} and L_{AB}^{13}
  – Positive error in L_{AB}^{12} usually => positive error in L_{AB}^{13}
- Weighted least squares is appropriate
  – "Stochastic model" is needed to derive weight matrix
  – Weight matrix is the inverse of the covariance matrix for the data:
  $W = C^{-1}$
- What is the covariance matrix for double differenced data, $C_{\nabla\Delta}$?

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Rule of propagation of errors

- In general, consider a vector X with covariance C_X
- What is the covariance for Y = KX ? (K is a matrix)
- Solution:

$C_X = \langle X X^T \rangle$ < > means "expected value"

$C_Y = \langle Y Y^T \rangle$ define: Y = KX
$= \langle KX (KX)^T \rangle$ recall: $(AB)^T = B^T A^T$
$= \langle KX X^T K^T \rangle$ $\langle K \rangle = K$ (constant)
$= K \langle X X^T \rangle K^T$

$C_Y = K C_X K^T$

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Double Difference Covariance

- Apply propagation of errors to double differencing
  – We can write $\nabla\Delta L = D L$
  – Where D is a matrix with elements of 1, –1, or 0
  Number of columns = number of observations
  Number of rows = number of independent double differences
  – Double differenced data covariance:
  $C_{\nabla\Delta} = D C D^T$
  ñ Hence, weight matrix for double differenced data is:
  $W_{\nabla\Delta} = (D C D^T)^{-1}$
  – For C, use a diagonal matrix, assuming a value for the standard deviation of an observation
    • Assumes observation errors are uncorrelated
    • Realistic value for observation error ~ few mm

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High Precision Software

- Developed by universities and institutions.
- Capabilities:
  – relative positioning at the level of few x 10^{-9} L
  – absolute (global) positioning at the level of 1 cm
  – orbit integration with appropriate force models
  – reliable data editing (cycle-slips, outliers)
  – accurate observation model (Earth model, media delay...)
  – estimation of all coordinates, orbits, tropospheric bias, receiver clock bias, polar motion, and Earth spin rate
  – note that "network adjustment" is not necessary
  – reliable ambiguity resolution using bootstrapping
  – either double-differencing, or clock-estimating
  – some have filtering capability (troposphere, clock, forces)

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Software Packages

- There are 3 ultra high-precision software packages which are widely used around the world by researchers:
  – BERNESE
    • Astronomical Institute, University of Berne, Switzerland
  – GAMIT
    • MIT, USA
  – GIPSY
    • Jet Propulsion Laboratory, Caltech, USA
- There are several other packages, but tend to be limited to the institutions that wrote them.

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Sources of Data & Information

- IERS: International Earth Rotation Service
  - Central Bureau: Paris Observatory, France
  - IERS Standards for models
  - IERS Annual Reports
  - International Terrestrial Reference Frame
  - Routine publication of Earth rotation parameters
- IGS: International GPS Service for Geodynamics
  - Central Bureau: Jet Propulsion Laboratory, USA
  - IGS Standards for permanent GPS stations
  - Operates global GPS network (~50 stations)
  - Distributes tracking data and precise ephemerides
  - Maintains on-line database with Internet access

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Error Sources

Satellite:	ephemeris, clock, S/A, and A/S
Propagation:	ionosphere, troposphere, multipath
Receiver:	antenna, clock, measurement error
Earth:	Earth surface kinematics
Processing:	cycle-slips, ambiguity
Stochastic:	errors in the error model

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Processing Overview

- Format data: T_A, L1, L2, P1, P2
  - Find cycle slips using data combinations
  - Form ionosphere-free combinations
- Generate orbit model
  - Either orbit integration using nominal initial conditions,
  - or use precise ephemerides
- Compute functional model and form b vector:
  - use T_A to construct $T_{proper, A}$, $T_{coord, A}$, $T^{coord, j}$, $T^{proper, j}$, T^j
  - need nominal station/satellite positions
- Least squares estimation (or Kalman Filter)
  - estimate station coordinates, tropospheric parameters
  - possibly estimate orbit parameters (initial conditions)
  - Ambiguity resolution
  - Update model, iterate if necessary

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Modes of GPS Positioning

- Aim of this lecture:
  - To review and compare methods of static positioning, and introduce methods for kinematic positioning.
- Overview
  - Point positioning
  - Differential positioning
  - Relative positioning
  - Static versus kinematic positioning
  - Precise kinematic positioning

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Point Positioning

- Procedure
  - single receiver, pseudoranges from ≥ 4 satellites
  - use satellite ephemerides to compute for each satellite:
    - 3 satellite coordinates and 1 clock bias
  - estimate using least squares
    - 3 station coordinates and 1 receiver clock bias
- Real-time point positioning
  - Broadcast Ephemerides from Navigation Message
    - Coordinate system: WGS-84

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Point Positioning

- Precise point positioning (post-processed)
  - IGS orbits and satellite clocks
    - (International GPS Service for Geodynamics)
  - Eliminates S/A: sub-metre positioning possible
    - ionosphere: use dual-frequency data combination
    - troposphere: modelled
    - multipath: reasonable environment
  - Using dual frequency carrier phase and pseudorange for few hours allows for sub-decimetre precision
    - Estimated parameters: 3 receiver coordinates, 1 receiver clock, 1 troposphere (at zenith), and 1 carrier phase bias for each satellite
    - JPL Orbit and clocks (Jet Propulsion Laboratory) with GIPSY-OASIS II software gives 1 cm precision

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Differential positioning

- Procedure
  - base station(s) tracking ≥ 4 satellites, computes and transmits "pseudorange corrections"
  - mobile receiver, corrects pseudoranges for ≥ 4 satellites
  - use broadcast ephemerides for orbits and sat. clocks
  - estimate using least squares, station position and clock
- Real-time differential positioning
  - Typical precision 1 to 10 metres
  - S/A, satellite errors, and propagation errors mitigated by this procedure
  - Errors can grow with distance to base station (e.g., ionosphere, troposphere)
  - Errors due to "age of correction" (several seconds)
  - Errors from pseudorange multipath, measurement error
  - Receiver can transform WGS-84 into national systems

Relative positioning

- Procedure
  - 2 stations (baseline), or multiple stations (network)
  - carrier phases from ≥ 4 satellites, then double-difference
  - use broadcast or IGS satellite orbits and clocks
  - assume values for one station and its clock time
    - e.g., use point position, or control point (WGS-84)
  - estimate, using weighted least squares, station coordinates, and carrier phase ambiguities
    - fix ambiguities to integer values and iterate.
- Post-processed relative positioning
  - Achievable precision: < 1 cm
    - over few × 10 km using broadcast orbits
    - over all distances using IGS orbits
  - Accuracy depends on hardware and software
    - dual frequency data, modelling capability, etc...

"Static" versus "Kinematic" Positioning

- Static Positioning
  - Stationary receivers
  - Use all data to estimate each station position as a constant
  - Can use any method described previously
- Kinematic Positioning
  - Mobile receivers
  - GPS positioning computation is identical to static problem
  - Solve for position at every epoch (e.g., 1 per second)
    - can just use current data (at that epoch)
    - can also use Kalman filter (= weighted average of position using current data + predicted position)
    - GPS can be integrated with other data types (gyrocompass, odometer, accelerometer, map info..)

Precise Kinematic Positioning

- Use "relative positioning" to a base station

$$\nabla\Delta L = \nabla\Delta\rho - \lambda_0\nabla\Delta N$$

  - double differenced carrier phase data, $\nabla\Delta L$
  - important: ambiguities $\nabla\Delta N$ must be known
  - "Initialisation" is the problem of finding $\nabla\Delta N$ in advance
  - Then we have:

$$\nabla\Delta L' = (\nabla\Delta L + \lambda_0\nabla\Delta N) = \nabla\Delta\rho$$

  - Can be done in real time if there is a radio link

Real Time Kinematic GPS

- "RTK GPS" similar to "Differential GPS"
  - Both use radio transmissions from a base station
    - "differential GPS" ⇒ pseudorange corrections
    - "RTK GPS" ⇒ dual frequency phase data, L
  - RTK requires FM radio link
    - higher data rate required than for differential GPS
    - limit on radio transmitter power due to legal restrictions
    - short range (15 km)
  - RTK must find correct values for $\nabla\Delta N$
    - More difficult if receiver is moving
    - "On the fly" ambiguity resolution
    - Range limited by effect of ionosphere on finding $\nabla\Delta N$

Initialisation Methods

- Method 1:
  - First do static positioning until $\nabla\Delta N$ are all resolved

$\nabla\Delta L = \nabla\Delta\rho - \lambda_0\nabla\Delta N$ > 15 min (?)

- Method 2:
  - First place 2 receivers at known coordinates
  - Solve for $\nabla\Delta N$:

$(\nabla\Delta L - \nabla\Delta\rho) = -\lambda_0\nabla\Delta N$ instantaneous

- Method 3:
  - Quickly swap 2 antennas between 2 fixed points, A and B
  - Solve for $\nabla\Delta N$

Before swap: $\nabla\Delta L_{AB} = \nabla\Delta\rho_{AB} - \lambda_0\nabla\Delta N$

After swap: $\nabla\Delta L_{BA} = \nabla\Delta\rho_{BA} - \lambda_0\nabla\Delta N = -\nabla\Delta\rho_{AB} - \lambda_0\nabla\Delta N$

(1) + (2): $\nabla\Delta L_{AB}\mathbf{(1)} + \nabla\Delta L_{BA}\ \mathbf{(2)} = -\mathbf{2}\lambda_0\nabla\Delta N$

Types of "Kinematic" Positioning

- True Kinematic
- Semi-Kinematic ("Stop and Go")
  - while tracking, stop at various fixed points
  - need to keep lock on signal
  - or correct for cycle slips (additional sensors may help)
- Pseudo-Kinematic
  - revisit fixed points within 1 hour
  - no need to use data while on the move
- Rapid Static
  - visit points only once
  - no need to use data while on the move
  - not really "kinematic"
  - rapid ambiguity resolution techniques needed
  - RTK systems often used in this mode

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